

For some function f and some non-zero number a , the derivative of f at a is given by

SCORE: _____ / 20 PTS

$$\lim_{h \rightarrow 0} \frac{2^{\csc[\pi(h+\frac{1}{2})]} - 2}{h}$$

- [a] Find f and a . Show that your answers are correct using the definition of the derivative at a point.

④ $f(x) = 2^{\csc \pi x}$, $a = \frac{1}{2}$ ④

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(\frac{1}{2}+h) - f(\frac{1}{2})}{h} = \boxed{\lim_{h \rightarrow 0} \frac{2^{\csc \pi(\frac{1}{2}+h)}}{h} - 2^{\csc \frac{\pi}{2}}}$$
$$= \lim_{h \rightarrow 0} \frac{2^{\csc \pi(h+\frac{1}{2})} - 2}{h}$$

- [b] Find the value of the limit, by evaluating $f'(a)$.

③ $f'(x) = 2^{\csc \pi x} \cdot \ln 2 \cdot -\csc \pi x \cot \pi x \cdot \pi$ ②

$$= 2^{\csc \frac{\pi}{2}} \cdot \ln 2 \cdot -\csc \frac{\pi}{2} \cot \frac{\pi}{2} \cdot \pi$$
$$= 2 \cdot \ln 2 \cdot -1 \cdot 0 \cdot \pi = \boxed{0} \text{ ②}$$

Prove the derivative of $\tan x$ using the definition of the derivative function. Show all steps.

SCORE: ____ / 15 PTS

Do NOT use the derivative shortcuts (such as the product rule etc.).

You may use the value of the two trigonometric limits proved in lecture without proving them again.

$$\begin{aligned}\frac{d}{dx} \tan x &= \frac{d}{dx} \frac{\sin x}{\cos x} = \lim_{h \rightarrow 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h} \quad (4) \\ &= \lim_{h \rightarrow 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h \cos(x+h)\cos x} \quad (4) \\ &= \lim_{h \rightarrow 0} \frac{\sin(x+h-x)}{h} \cdot \lim_{h \rightarrow 0} \frac{1}{\cos(x+h)\cos x} \quad (5) \\ &= 1 \cdot \frac{1}{\cos^2 x} = \sec^2 x \quad (2)\end{aligned}$$

If g is a function and $f(x) = x^2 g(\frac{1}{x})$, find a formula for $f''(x)$, which may involve g , g' and/or g'' .

SCORE: _____ / 20 PTS

$$= x^2 g(x^{-1})$$

$$f'(x) = 2xg(x^{-1}) + x^2 g'(x^{-1})(-x^{-2})$$

$$\hat{=} \underline{2xg(x^{-1})} - \underline{g'(x^{-1})} \textcircled{5}$$

$$f''(x) = 2g(x^{-1}) + 2xg'(x^{-1})(-x^{-2}) - g''(x^{-1})(-x^{-2})$$

$$\hat{=} \underline{2g(\frac{1}{x})} - \underline{\frac{2}{x}g'(\frac{1}{x})} + \underline{\frac{1}{x^2}g''(\frac{1}{x})} \textcircled{5} \textcircled{5}$$

Prove that $y = mx + b$ and $x^2 + y^2 - 2by = c$ are orthogonal trajectories.

SCORE: _____ / 20 PTS

$$\frac{dy}{dx} = m \quad (1)$$

$$2x + 2y \frac{dy}{dx} - 2b \frac{dy}{dx} = 0 \quad (6)$$

$$\frac{dy}{dx} = \frac{2x}{2b-2y} = \frac{x}{b-y} \quad (3)$$

$$m \cdot \frac{x}{b-y} = \left| \frac{mx}{b-y} \right| = \left| \frac{y-b}{b-y} \right| = -1 \quad (4) \quad (3)$$

Let $f(x) = (1 + \ln x)^{\ln x}$. $f(e) = (1+1)^1 = 2$

SCORE: ____ / 25 PTS

- [a] If x changes from e to 3, find dy .

(3) $\ln y = \ln x \times \ln(1 + \ln x)$

(3) $\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \ln(1 + \ln x) + \ln x \cdot \frac{1}{1 + \ln x} \cdot \frac{1}{x}$ (5)

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \left(\ln(1 + \ln x) + \frac{\ln x}{1 + \ln x} \right)$$

$$\frac{1}{2} \frac{dy}{dx} \Big|_{x=e} = \frac{1}{e} \left(\ln(1+1) + \frac{1}{1+1} \right) = \frac{1}{e} \left(\ln 2 + \frac{1}{2} \right), \text{ (3)}$$

$$\frac{dy}{dx} \Big|_{x=e} = \frac{2}{e} \left(\ln 2 + \frac{1}{2} \right), \text{ (2)}$$

$$dy = \frac{2}{e} \left(\ln 2 + \frac{1}{2} \right) \Delta x$$

$$= \frac{2}{e} \left(\ln 2 + \frac{1}{2} \right) (3 - e), \text{ (3)}$$

- [b] Approximate $f(3)$ using your answer to part [a].

$$f(3) \approx f(e) + dy$$

$$= 2 + \frac{2}{e} \left(\ln 2 + \frac{1}{2} \right) (3 - e) \text{ (3)}$$

Find $\frac{d^3}{dx^3} \arctan \frac{1}{x^2}$.

SCORE: ___ / 25 PTS

HINT: Simplify often.

$$\frac{d}{dx} \arctan \frac{1}{x^2} = \left| \frac{1}{1 + \left(\frac{1}{x^2}\right)^2} \right| \cdot \left| -\frac{2}{x^3} \right| = -\frac{2}{x^3 + \frac{1}{x}} = \left| -\frac{2x}{x^4 + 1} \right|$$

$$\frac{d^2}{dx^2} \arctan \frac{1}{x^2} = -\frac{2(x^4+1) - 2x(4x^3)}{(x^4+1)^2} = \frac{6x^4 - 2}{(x^4+1)^2}$$

$$\frac{d^3}{dx^3} \arctan \frac{1}{x^2} = \frac{24x^3(x^4+1)^2 - (6x^4-2)2(x^4+1)4x^3}{(x^4+1)^4} \quad (4)$$

$$= \frac{8x^3(3x^4+3-6x^4+2)}{(x^4+1)^3}$$

$$= \left| \frac{8x^3(5-3x^4)}{(x^4+1)^3} \right| \quad (3)$$

The base of a 16 foot tall conical water tank has a radius of 12 feet.

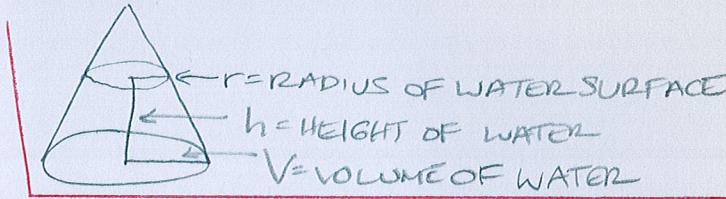
SCORE: _____ / 25 PTS

Water is draining from the tank at 24π cubic feet per minute.

At the moment when the water is 8 feet high in the tank, is the radius of the surface of the water expanding or shrinking, and at what rate?

You must state/show clearly what each variable you use represents.

You must show the units during the intermediate steps of your work, and you must state the units for the final answer.



$$\frac{dV}{dt} = -24\pi \frac{\text{FT}^3}{\text{MIN}}, \text{ WANT } \left. \frac{dr}{dt} \right|_{h=8 \text{ FT}}$$

$$V = \frac{1}{3}\pi (12 \text{ FT})^2 (16 \text{ FT}) - \frac{1}{3}\pi r^2 (16 \text{ FT} - h)$$

$$V = \frac{1}{3}\pi (12 \text{ FT})^2 (16 \text{ FT}) - \frac{1}{3}\pi r^2 \left(\frac{4}{3}r\right) = \frac{1}{3}\pi (12 \text{ FT})^2 (16 \text{ FT}) - \frac{4}{9}\pi r^3 \quad (7)$$

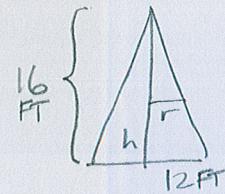
$$\frac{dV}{dt} = -\frac{4}{3}\pi r^2 \frac{dr}{dt} \quad (4)$$

$$-24\pi \frac{\text{FT}^3}{\text{MIN}} = -\frac{4}{3}\pi (6 \text{ FT})^2 \left. \frac{dr}{dt} \right|_{h=8 \text{ FT}} \quad (4)$$

$$-24\pi \frac{\text{FT}^3}{\text{MIN}} = -48\pi \text{ FT}^2 \left. \frac{dr}{dt} \right|_{h=8 \text{ FT}}$$

$$\frac{1}{2} \frac{\text{FT}}{\text{MIN}} = \left. \frac{dr}{dt} \right|_{h=8 \text{ FT}}$$

(3)



$$\frac{16 \text{ FT} - h}{r} = \frac{16 \text{ FT}}{12 \text{ FT}}$$

$$16 \text{ FT} - h = \frac{4}{3}r$$

$$\text{WHEN } h = 8 \text{ FT}$$

$$\frac{4}{3}r = 8 \text{ FT} \rightarrow r = 6 \text{ FT}$$

THE RADIUS OF THE WATER'S SURFACE IS EXPANDING (2)

BY $\frac{1}{2} \frac{\text{FT}}{\text{MIN}}$ (2) ← MUST BE SHOWN IN WORK